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ON THE TORSION OF A RIGID-PLASTIC CYLINDER

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A foundation is given for the formula for the limit moment for a certain class of anisotropic inhomogeneous rigid-plastic media in the problem of torsion of a cylinder with an arbitrary cross section. A formula is derived for the rate of warping of the cross section, and a uniqueness theorem is proved in the case of a simply-connected cross section.

1. Formulation of the problem. A rigid-plastic medium [1] is determined by its dissipative potential $\varphi(x, e_{ij})$ [2], the positive, semi-additive, positively-homogeneous first degree function e_{ij} which is independent of the trace of the matrix e_{ij}.

Let external forces with the volume density f_v and surface density f_s act on a medium in a volume ω. One of the most important quantities in the theory of a rigid-plastic medium is the limit load coefficient c_* for a given system of external forces, defined by the formula

$$\frac{1}{c_*} = \sup_{\mathbf{u}} \left[\left(\int_\omega \mathbf{f}_v \mathbf{u}\, d\omega + \int_{\partial\omega} \mathbf{f}_s \mathbf{u}\, dS \right) \times \right. \tag{1.1}$$

$$\left. \left(\int_\omega \varphi(x, e_{ij}(x))\, d\omega \right)^{-1} \right], \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where $\partial\omega$ is the boundary of ω. The vector-functions $\mathbf{u}(x)$ in (1.1) form a linear space, are solenoidal, and satisfy definite boundary conditions.

Let $\varphi^\circ(x, \sigma_{ij})$ denote the polar for $\varphi(x, e_{ij})$ [3], i.e. a function possessing the properties: $\varphi^\circ(x, \sigma_{ij})$ is a positive, semi-additive, positively homogeneous function of the first degree in σ_{ij} (the matrices σ_{ij} are assumed to have zero trace), where for all σ_{ij}, e_{ij}

$$\varphi^\circ(x, \sigma_{ij})\, \varphi(x, e_{ij}) \geqslant \sum_{ij} \sigma_{ij} e_{ij} \tag{1.2}$$

For any $e = (e_{ij})$ there exist $\sigma_{ij}(e), \sum_{ij} \sigma_{ij}^2(e) > 0$, such that the equality holds in (1.2).

It can be shown that the yield condition for a rigid-plastic medium is given by the equality $\varphi^\circ(x, \sigma_{ij}) = 1$.

Formulation of the problem of torsion of a rigid-plastic bar is analogous to the corresponding formulation of the problem for a linearly elastic bar [4] and consists of the following. Let ω be a cylinder

$$\omega = D \times [0, H], \quad (x_1, x_2) \in D, \quad 0 \leqslant x_3 \leqslant H$$

The boundary conditions on $\mathbf{u}(x)$ are assumed to have the form:

$$\mathbf{u}(x_1, x_2, 0) = (0, 0, g_1(x_1, x_2)), \ \mathbf{u}(x_1, x_2, H) = (-\alpha x_2, \alpha x_1\, g_2(x_1, x_2))$$

where g_1, g_2 are arbitrary functions and α is an arbitrary real number. The external forces act only on the endface $x_3 = H$ and $\mathbf{f}_s = (-x_2, x_1, 0)$.

Let c_* be the limit load coefficient in the problem of torsion (the limit moment). Let $[\sigma_{ij}]$ be a system of functions in which σ_{13}, σ_{23} are arbitrary, while the remaining σ_{ij} equal zero. Let $\lambda\ (x_1, x_2)$ be a smooth function $\lambda \mid_{\partial D} = \text{const}$. Let $[\sigma_{ij}]\ (\lambda)$ be a system of functions $[\sigma_{ij}]$ in which $\sigma_{13} = \partial\lambda / \partial x_2$, $\sigma_{23} = -\partial\lambda / \partial x_1$. Then if $\varphi^\circ\ (x, [\sigma_{ij}]\ (\lambda)) \leqslant 1$, then $c_* \geqslant A_1 / A_2$

$$A_1 = -\int\limits_D \left(x_1 \frac{\partial\lambda}{\partial x_1} + x_2 \frac{\partial\lambda}{\partial x_2}\right) d\mu, \quad A_2 = \int\limits_D (x_1^2 + x_2^2)\, d\mu$$

We consider the vector functions $\mathbf{U} = (-x_2x_3,\ x_1x_3,\ u\ (x_1, x_2))$. The $\mathbf{U}$ evidently enter into the class of admissible $\mathbf{u}\ (x)$. Setting $2e_{13} = \partial u / \partial x_1 - x_2$, $2e_{23} = \partial u / \partial x_2 + x_1$ and assuming that $\varphi\ (x, e_{ij}) = \varphi\ (x_1, x_2, e_{ij})$, we find

$$\sup_{\lambda\in\Lambda} A_1 \leqslant c_* A_2 \leqslant \inf_u \int\limits_D \varphi\, (x_1, x_2, [e_{ij}])\, d\mu \tag{1.3}$$
$$\Lambda : \lambda|_{\partial D} = \text{const}, \quad \varphi^\circ\ (x_1, x_2,\ [\sigma_{ij}]\ (\lambda)) \leqslant 1$$

2. Calculation of the limit moment. We assume that $\varphi^\circ\ (x_1, x_2, [\sigma_{ij}])$ is polar for $\varphi\ (x_1, x_2, [e_{ij}])$. This condition will be satisfied if

$$\inf_{e'} \varphi\ (x_1, x_2, e_{ij}) = \varphi\ (x_1, x_2, [e_{ij}]), \quad e' = \{e_{11}, e_{12}, e_{22}\}, \quad e_{33} = -e_{11} - e_{22}$$

Let $\mathbf{e} = (e_1, e_2)$ from R^2. We set $\psi\ (x_1, x_2, \mathbf{e}) = \varphi\ (x_1, x_2, [e_{ij}])$, where $e_1 = 2e_{13}$, $e_2 = 2e_{23}$. Then $\psi^\circ\ (x_1, x_2, \boldsymbol{\sigma}) = \varphi^\circ\ (x_1, x_2, [\sigma_{ij}])$, where $\sigma_1 = \sigma_{13}$, $\sigma_2 = \sigma_{23}$ ($\varphi^\circ\ (x_1, x_2, \boldsymbol{\sigma})$ is polar for $\psi\ (x_1, x_2, \mathbf{e})$). Let us rewrite (1.3) in the form

$$\sup_{\lambda\in\Lambda} A_1 \leqslant c_* A_2 \leqslant \inf_u \int\limits_D \psi\, (x_1, x_2, \nabla u - \mathbf{t}_0)\, d\mu \tag{2.1}$$
$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)$$
$$\mathbf{t}_0 = (x_2, -x_1), \quad \Lambda : \lambda\,|_{\partial D} = \text{const}, \quad \psi^\circ (x_1, x_2, G\lambda) \leqslant 1$$
$$G\lambda = \left(\frac{\partial\lambda}{\partial x_2}, -\frac{\partial\lambda}{\partial x_1}\right)$$

Let $\overline{\Lambda}$ denote the set $\lambda\ (x_1, x_2)$, for which

$$\lambda\ |_{\partial D} = \text{const}, \quad \operatorname{vrai\,max} \psi^\circ\ (x_1, x_2, G\lambda) \leqslant 1$$

Theorem 1. If D is a bounded domain with a sufficiently smooth boundary, $\psi\ (x_1, x_2, \mathbf{e})$ is a sufficiently smooth function of its arguments for $|\ \mathbf{e}\ | > 0$ and $\mathbf{t}\ (x_1, x_2)$ is a sufficiently smooth vector-function, then

$$\inf_u \int\limits_D \psi\, (x_1, x_2, \nabla u - \mathbf{t})\, d\mu = \sup_{\lambda\in\overline{\Lambda}} - \int\limits_D \mathbf{t} G\lambda\, d\mu \tag{2.2}$$

Proof. Let $\Phi_\varepsilon\ (\xi)$ be an infinitely differentiable convex function of ξ, $\xi \geqslant 0$, where $\Phi_\varepsilon\ (\xi) = \xi$ for $\xi \geqslant \varepsilon$, $\Phi_\varepsilon'\ (\xi) \leqslant 1$, $\Phi_\varepsilon^{(k)}\ (0) = 0$, $k = 1, 2, \ldots$. We set $\psi_\varepsilon\ (x_1, x_2, \mathbf{e}) = \Phi_\varepsilon\ (\psi\ (x_1, x_2, \mathbf{e}))$. Then

$$0 \leqslant \psi_\varepsilon\, (x_1, x_2, \mathbf{e}) - \psi\, (x_1, x_2, \mathbf{e}) \leqslant \varepsilon \tag{2.3}$$
$$\mathbf{e}\nabla_e \psi_\varepsilon\ (x_1, x_2, \mathbf{e}) \leqslant \psi\ (x_1, x_2, \mathbf{e}) \leqslant \varepsilon + \mathbf{e}\nabla_e\psi_\varepsilon\ (x_1, x_2, \mathbf{e})$$

Let us consider the functional

$$I_\varepsilon(v) = \int\limits_G \left[\frac{\varepsilon}{2}(\nabla v - \mathbf{t})^2 + \psi_\varepsilon(x_1, x_2, \nabla v - \mathbf{t})\right] d\mu$$

Let u_ε minimize $I_\varepsilon(v)$. This is a function [5, 6] from $C_2(\bar{D})$ and it satisfies the relationships

$$\operatorname{div} \mathbf{Q}_\varepsilon = 0, \quad \mathbf{Q}_\varepsilon \mathbf{n} \mid_{\partial D} = 0 \tag{2.4}$$

$$\mathbf{Q}_\varepsilon = \varepsilon(\nabla u_\varepsilon - \mathbf{t}) + \nabla_e \psi_\varepsilon(x_1, x_2, \nabla u_\varepsilon - \mathbf{t}), \quad \mathbf{n} \text{ is normal to } \partial D$$

It follows from (2.4) that

$$\mathbf{Q}_\varepsilon = G\lambda_\varepsilon \tag{2.5}$$

We find from the second equality in (2.4) and (2.5) the $\lambda_\varepsilon \mid_{\partial D} = \text{const}$. Moreover as $\varepsilon \to 0$

$$\int\limits_D \varepsilon^2 (\nabla u_\varepsilon - \mathbf{t})^2 \, d\mu \to 0 \tag{2.6}$$

$$\psi^\circ(x_1, x_2, \varepsilon(\nabla u_\varepsilon - \mathbf{t})) + 1 \geqslant \psi^\circ(x_1, x_2, G\lambda_\varepsilon)$$

By virtue of (2.6), the λ_ε are uniformly bounded with respect to ε in $W_2{}^1(D)$ and an λ_0 exists from $W_2{}^1(D)$ (weak limit point λ_ε) such that

$$\lambda_0 \mid_{\partial D} = \text{const}, \quad \int\limits_{D'} d\mu \geqslant \int\limits_{D'} [\psi^\circ(x_1, x_2, G\lambda_0)]^2 \, d\mu, \quad D' \subseteq D$$

Therefore, $\operatorname{vrai\,max} \psi^\circ(x_1, x_2, G\lambda_0)^- \leqslant 1$.

We find from (2.5) and (2.3)

$$\int\limits_D (\nabla u_\varepsilon - \mathbf{t}) G\lambda_\varepsilon \, d\mu + \varepsilon \operatorname{mes} D \geqslant I_0(u_\varepsilon), \quad I_0(v) = I_\varepsilon(v)\mid_{\varepsilon=0}$$

Hence, the inequality

$$-\int\limits_D \mathbf{t} G\lambda_0 \, d\mu \geqslant \inf_u \int\limits_D \psi(x_1, x_2, \nabla u - \mathbf{t}) \, d\mu \tag{2.7}$$

follows.

The assertion of Theorem 1 follows from the inequalities (2.1) and (2.7).

Theorem 1 admits of strengthening. Namely, if D is a finitely connected domain with piecewise-smooth boundary, and $\psi(x_1, x_2, \mathbf{e})$ is a continuous function of its arguments, then (2.2) is valid, and hence $\bar{\Lambda}$ can be replaced by Λ.

Let us note that c_* yields the lower bound of the limit moment in the problem of constrained torsion, i.e. for boundary conditions of the form

$$\mathbf{u}(x_1, x_2, 0) = (0, 0, 0), \quad \mathbf{u}(x_1, x_2, H) = (-\alpha x_2, \alpha x_1, 0)$$

3. Formulas for the warping rate of the cylinder transverse section. We obtain a formula for $u_0(x_1, x_2)$, which minimizes $I_0(v)$. It is evident that

$$\inf_v I_0(v) \geqslant \inf_{\mathbf{p}} \int\limits_D [\psi(x_1, x_2, \mathbf{p} - \mathbf{t}) - \mathbf{p} G\lambda_0] \, d\mu \tag{3.1}$$

where λ_0 (the stress function [1]) satisfies the following conditions almost everywhere in D : $\lambda_0 \mid_{\partial D} = \text{const}$, $\psi^\circ(x_1, x_2, G\lambda_0) = 1$; $\lambda_0(x_1, x_2)$ is a continuous function. Moreover, certain conditions will still be imposed on λ_0.

The values of $\mathbf{p}$ for which the integrand in the right side of (3.1) reaches its least value are determined from the system

$$\nabla_p \psi(x_1, x_2, \mathbf{p} - \mathbf{t}) = G\lambda_0$$

Therefore, if a function u_0 satisfying the overdefined system of equations

$$\nabla_p \psi (x_1, x_2, \Delta u_0 - \mathbf{t}) = G\lambda_0 \tag{3.2}$$

is found, then $u_0 (x_1, x_2)$ minimizes $I_0 (v)$.

Let us assume that $\psi (x_1, x_2, \mathbf{e})$, $\psi^\circ (x_1, x_2, \mathbf{e})$ be smooth functions of their arguments for $| \mathbf{e} | > 0$. It hence follows that for fixed x_1, x_2 there exists a unique vector $\mathbf{e} (\mathbf{q})$ for any $\mathbf{q}$ such that $\nabla_e \psi (x_1, x_2, \mathbf{e} (\mathbf{q})) \parallel \mathbf{q}$ (parallel).

Lemma 1. The relationship $\psi (x_1, x_2, \mathbf{e}) \psi^\circ (x_1, x_2, \mathbf{q}) = \mathbf{eq}$ is satisfied if and only if $\nabla_e \psi (x_1, x_2, \mathbf{e}) \parallel \mathbf{q}$.

The assertion of the lemma results from the geometric properties of the scalar product.

Corollary 1. The relationships $\nabla_e \psi (x_1, x_2, \mathbf{e}) \parallel \mathbf{q}$ and $\nabla_q \psi^\circ (x_1, x_2, \mathbf{q}) \parallel \mathbf{e}$ are equivalent.

Corollary 2. For any

$$\mathbf{e} (| \mathbf{e} | > 0) : \mathbf{e} = \psi (x_1, x_2, \mathbf{e}) \nabla_q \psi^\circ (x_1, x_2, \mathbf{q}) |_{\mathbf{q} = \nabla_e \psi (x_1, x_2, \mathbf{e})}$$

Lemma 2. The systems of equations $\psi^\circ (x_1, x_2, \mathbf{q}) \nabla_e \psi (x_1, x_2, \mathbf{e}) = \mathbf{q}$, $\psi (x_1 x_2, \mathbf{e}) \nabla_q \psi^\circ (x_1, x_2, \mathbf{q}) = \mathbf{e}$ ($\mathbf{q}$ is a given vector) are equivalent and solvable uniquely to the accuracy of a factor.

Proof. The equivalence of the system results from Corollary 2, and their unique solvability follows from the properties of $\psi (x_1, x_2, \mathbf{e})$.

Thus, by virtue of Lemma 2 the system (3.2) is equivalent to the system

$$\nabla u_0 - \mathbf{t} = \psi (x_1, x_2, \nabla u_0 - \mathbf{t}) \nabla_q \psi^\circ (x_1, x_2, G\lambda_0) \tag{3.3}$$

Let us first consider a doubly-connected domain D bounded by piecewise-smooth contours Γ_1, Γ_2 (Γ_2 lies inside Γ_1). Simply connected domains when Γ_2 shrinks to a point are a particular case of such domains. Let λ_0 be a smooth function in D, except perhaps for a finite number of smooth curves, and

$$\lambda_0 \geqslant 0, \ \ \lambda_0 |_{\Gamma_1} = 0, \ \ \lambda_0 |_{\Gamma_2} = \text{const}, \ \ \psi^\circ (x_1, x_2, G\lambda_0) = 1$$

We call a set of these smooth curves and Γ_2 a set of singularities Γ.

Let us consider a field of directions $\mathbf{v}$ and D

$$\mathbf{v} = -G\psi^\circ (x_1, x_2, G\lambda_0) \ / \ | G\psi^\circ (x_1, x_2, G\lambda_0)|$$

Let $\mathbf{l}$ be the unit tangent vector to the level line $\lambda_0 = c$ which gives the counterclockwise bypass of the domain $\lambda_0 \geqslant c$. Since $(\nabla\lambda_0, \bar{\mathbf{v}}) > 0$ in $D \setminus \Gamma$, then the level lines λ_0 and the integral curves of the field $\mathbf{v}$ form a regular network of lines in $D \setminus \Gamma$. Let s be the arclength on Γ_1 measured from a certain point on Γ_1 in the direction $\mathbf{l}$, $0 \leqslant s \leqslant L$. We fix a point s on Γ_1 and we introduce a parameter n on the integral curve of the field $\mathbf{v}$ passing through the point. Namely, $n = \lambda_0 (x_1, x_2)$ will correspond to the point (x_1, x_2) on the integral curve. It follows from the condition $(\nabla\lambda_0, \mathbf{v}) > 0$ that different values of n, $0 \leqslant n \leqslant n (s)$ correspond to different points of the integral curve. With respect to λ_0 it is assumed that an integral curve of the field $\mathbf{v}$ issuing from Γ_1 passes through each point $D \setminus \Gamma$. Therefore we have the system

$$x_1 = x_1 (s, n), \quad x_2 = x_2 (s, n) \tag{3.4}$$

which governs the integral trajectory of the field $\mathbf{v}$ for a fixed s and governs the level line λ_0 parametrized by s for fixed n, when s is from the domain of definition (3.4).

Let the domain of variation of s for fixed n consist of a finite number of segments $s_i^n \leqslant s \leqslant s_{i+1}^n$, $i = 1, \ldots, r(n)$, where $s_i^n < s < s_{i+1}^n$ yields a mapping in $D \setminus \Gamma$ or in Γ, (s_i^n, n) is a point of Γ. Let us introduce natural parametrization of a piece of level line $\lambda_0 = n$ in each interval $s_i^n < s < s_{i+1}^n$: $\sigma = \sigma(s, n)$, $\partial\sigma / \partial s \geqslant 0$.

Thus,(3.4) determines the curvilinear coordinates in D and

$$\frac{\partial}{\partial \nu} = (\nabla\lambda_0, \mathbf{v}) \frac{\partial}{\partial n}, \quad \frac{\partial}{\partial l} = \frac{1}{\partial\sigma / \partial s} \frac{\partial}{\partial s}$$

It can be shown that the system (3.3) is equivalent to the relationships

$$\partial u_0 / \partial l \geqslant (\mathbf{t}, \mathbf{l}) \quad \partial u_0 / \partial \nu = (\mathbf{t}, \mathbf{v})$$

from which it follows that

$$u_0(s, n) = \int_0^n \frac{\mathbf{t}(s, \mu)\, \mathbf{v}(s, \mu)}{\nabla\lambda_0(s, \mu)\, \mathbf{v}(s, \mu)} d\mu + q(s), \quad q(L) - q(0) \leqslant 0 \tag{3.5}$$

$$q'(s) \geqslant \frac{\partial\sigma}{\partial s} \mathbf{tl} - \int_0^n \frac{\partial}{\partial s} \frac{\mathbf{tv}}{\nabla\lambda_0\mathbf{v}} d\mu = A(s, n) \tag{3.6}$$

$$\frac{\partial A}{\partial n} = \frac{\partial\sigma}{\partial s} \frac{B}{\nabla\lambda_0\mathbf{v}} + \mathbf{t}\gamma, \quad B = \frac{\partial\mathbf{t}}{\partial\nu}\mathbf{l} - \frac{\partial\mathbf{t}}{\partial l}\mathbf{v}$$

$$\gamma = \frac{\partial}{\partial n}\left(\mathbf{l}\frac{\partial\sigma}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{\mathbf{v}}{\nabla\lambda_0\mathbf{v}}\right)$$

Lemma 3. The following equalities are valid:

$$\gamma = 0, \quad B = \left(\frac{\partial t_1}{\partial x_2} - \frac{\partial t_2}{\partial x_1}\right) \nabla\lambda_0\mathbf{v}$$

Proof. The first assertion of the lemma follows from the equalities

$$\mathbf{l} = \frac{\partial\mathbf{x}}{\partial\sigma} = \frac{\partial\mathbf{x}}{\partial s} \Big/ \frac{\partial\sigma}{\partial s}, \quad \mathbf{v} = \frac{\partial\mathbf{x}}{\partial\nu} = \frac{\partial\mathbf{x}}{\partial n}(\nabla\lambda_0, \mathbf{v}) \tag{3.7}$$

The second assertion is proved by passing to the coordinates (x_1, x_2) in the expression for B in (3.6).

Let us assume that

$$\partial t_1/\partial x_2 - \partial t_2/\partial x_1 \geqslant 0 \text{ in } D \tag{3.8}$$

Condition (3.8) is satisfied for $\mathbf{t}_0$ in (2.1). It follows from Lemma 3 and from the condition (3.8) that (3.6) is equivalent to the inequality

$$q'(s) \geqslant A(s, \quad n(s)) \tag{3.9}$$

Let us set $q(s) = Q(s) + p(s)$, where $p'(s) \geqslant 0$ and $Q'(s)$ equals the right side in (3.9). Then

$$u_0(s, n) = P_1(s, n) + P_2(s) + p(s) \tag{3.10}$$

$$P_1(s, n) = -\int_n^{\cdot} \frac{\mathbf{t}(s, \mu)\, \mathbf{v}(s, \mu)}{\nabla\lambda_0(s, \mu)\, \mathbf{v}(s, \mu)} d\mu$$

$$P_2(s) = \int_0^s \mathbf{t}(\alpha, n(\alpha)) \left(\frac{\partial\sigma(\alpha, n)}{\partial\alpha}\Big|_{n=n(\alpha)} \mathbf{l}(\alpha, n(\alpha)) + \frac{n'(\alpha)\, \mathbf{v}(\alpha, n(\alpha))}{\nabla\lambda_0(\alpha, n(\alpha))\, \mathbf{v}(\alpha, n(\alpha))} \right) d\alpha$$

We note that

$$P_1(s,n) = \int_{C_1} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau, \quad \mathbf{m}(\tau) = -\mathbf{v}(\tau)$$

where C_1 is a piece of the integral curve of the field $\mathbf{v}$ from the point (s, n) to $(s, n(s))$, $\boldsymbol{\tau}$ is the arclength along this curve. The curve Γ is given by the equation $\mathbf{x}(s, n(s)) = \mathbf{x}$. The single tangent vector $\mathbf{m}$ to Γ has the form

$$\mathbf{m} = \left(\frac{\partial \mathbf{x}}{\partial s} + \frac{\partial \mathbf{x}}{\partial n} n'(s)\right) \Big/ \left|\frac{\partial \mathbf{x}}{\partial s} + \frac{\partial \mathbf{x}}{\partial n} n'(s)\right|$$

Using (3.7), we obtain

$$P_2(s) = \int_{C_2} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau$$

where C_2 is the piece of Γ from $(0, n(0))$ to $(s, n(s))$, and τ is the arclength along C_2.

If follows from condition (3.5) that

$$p(L) - p(0) \leqslant -\int_{\Gamma} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau$$

where τ is the arclength on Γ which grows in the direction $\mathbf{m}$.

Let $\mathbf{t}(\tau)$ satisfy the condition

$$\int_{\Gamma} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau \leqslant 0 \tag{3.11}$$

Condition (3.11) is satisfied, for example, for the $\mathbf{t}_0$ in (2.1). Let us formulate the results obtained in the form of a theorem.

Theorem 2. Let

$$u_0(s,n) = \int_{C} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau + p(s) \tag{3.12}$$

where C consists of C_1 and C_2, $p(s)$ is a nondecreasing function, and

$$p(L) - p(0) \leqslant -\int_{\Gamma} \mathbf{t}(\tau)\,\mathbf{m}(\tau)\,d\tau$$

Then $u_0(s, n)$ minimizes $I_0(v)$.

Corollary 3. If D is a simply-connected domain, then $p(s) = \text{const}$ and $u_0(s, n)$ is a continuous, piecewise-smooth function.

Corollary 4. (Uniqueness theorem). Let $\varphi(x_1, x_2, e_{ij}) > \varphi(x_1, x_2, [e_{ij}])$ for $\sum_{i,j=1}^{2} e_{ij}^2 > 0$ and $\psi(x_1, x_2, \mathbf{e})$ a strictly convex function of $\mathbf{e}$. Then the torsion problem in a simply connected domain ω has a unique solution in the class of vector functions with integrable derivatives.

Proof. The following inequalities are valid:

$$\begin{aligned} &\frac{1}{H}\int_{\omega} \varphi(x_1, x_2, e_{ij})\,d\omega \geqslant \frac{1}{H}\int_{\omega} \varphi(x_1, x_2, [e_{ij}])\,d\omega \geqslant \\ &\int_{D} \psi\left(x_1, x_2, 2\frac{1}{H}\int_0^H e_{13}\,dx_3,\ 2\frac{1}{H}\int_0^H e_{23}\,dx_3\right) d\mu = \\ &I_0(v) = \int_{D} \psi(x_1, x_2, \nabla v - \mathbf{t}_0)\,d\mu \end{aligned} \tag{3.13}$$

$$v = \frac{1}{H}\int_0^H u_3\,dx_3, \quad \mathbf{u}(x_1x_2, 0) = (0, 0, u_3(x_1, x_2, 0))$$

$$\mathbf{u}(x_1, x_2, H) = (-Hx_2, Hx_1, u_3(x_1, x_2, H))$$

The equality sign holds in the relationships (3.13) if and only if $e_{11} = e_{12} = e_{22} = 0$ and e_{13}, e_{23} are proportional to the same function of x_3. Uniqueness to the accuracy of a constant component of the function minimizing $I_0(v)$ follows from the unique solvability of the system (3.2). Therefore, if $\mathbf{u}(x)$ is the solution of the torsion problem in the simply-connected domain ω, then it follows from the first inequality in (3.13) and the incompressibility condition that

$$u_1 = -c(x_3)\,x_2, \quad u_2 = c(x_3)\,x_1, \quad u_3 = u(x_1, x_2) \tag{3.14}$$

It follows from the second inequality in (3.13) that $c(x_3) = \text{const}$ if the function $u(x_1, x_2)$ in (3.14) is not a constant. Therefore, $c(x_3) = \text{const}$ if the domain D is not a circle. If D is a circle, then $u_3 = u(x_1, x_2) = \text{const}$ and (3.14) is a solution of the torsion problem for any monotonic function with $c(x_3)$, $c(0) = 0$.

Let us turn to an analysis of multiconnected domains D. We note that (3.12) contains even discontinuous functions if $p(s)$ is a discontinuous function (these discontinuous functions can be understood as the limits of smooth functions). Let us examine an arbitrary doubly-connected domain D bounded by the contours Γ_1, Γ_2 (Γ_2 lies within Γ_1). Let $\lambda_0(x_1, x_2)$ be a continuous function in D, such that $\lambda_0 \mid_{\Gamma_1} = 0$, $\lambda_0 \mid_{\Gamma_2} = \text{const}$ and $\psi^\circ(x_1, x_2, G\lambda_0) = 1$ almost everywhere in D. Let us consider the level line of $\Gamma_1' : \lambda_0 = c$, such that the contour Γ_2 is within Γ_1' and Γ_1', Γ_2 have common points. It is assumed that the function λ_0 has the above-mentioned singularity structure in the domain D' included between Γ_1 and Γ_1', and in the simply-connected domain D'' whose boundary consists of Γ_1', Γ_2. The rate of warping is found as follows. The function $u_0(s, n)$ is first constructed in D'' by means of (3.12). In this case $p(s) = \text{const}$. Later $u_0(s, n)$ is determined by (3.12) in D', and $p(s)$ is here selected so that $u_0(s, n)$ is continuous almost everywhere on Γ_1.

For instance, let D be the domain between the imbedded nonconcentric circles K_1, K_2, $K_1 \supset K_2$ and $\psi^\circ(x_1, x_2, G\lambda_0) = |\nabla\lambda_0|$. Then D' is a concentric ring and D'' is a simply-connected horseshoe-like domain bounded by two tangent circles. In this case $u_0(s, n)$ is determined uniquely and is a discontinuous function with a discontinuity going along the shortest segment connecting the boundaries K_1 and K_2. The magnitude of the discontinuity is determined in terms of the integral of the vector field t along the line of singularities in the horseshoe-shaped domain D''. If $t = t_0$, R is the radius of K_1, r the radius of K_2 and ρ the spacing between the boundaries K_1 and K_2, then the magnitude of the jump in the warping rate is

$$\pi(R - \rho + r)\sqrt{(R - \rho)\,r}$$

An analogous construction permits finding the rate of warping in multiconnected domains D.

It has been noted in Sect. 2 that c_* yields the lower bound for the limit moment in the constrained torsion problem. By using the expression found in Sect. 3 for the warping rate, it can be shown that for a simply-connected domain ω, the limit moment in the constrained torsion problem differs from c_* by the quantity P / H, where H is the cylinder height, and the quantity P is estimated in terms of the function $u_0(s, n)$ defined in (3.12).

4. Construction of the stress function. The function λ_0 was introduced in Sects. 2 and 3. Methods of constructing the function λ_0 will be given below. Let us first consider the case of a simply-connected domain D.

a) Let $\psi^\circ(x_1x_2, \mathbf{e}) = |\mathbf{e}|$. Then the function $\lambda_0(x_1, x_2)$ can be found as follows. We construct a family of circular cones with semi-apex angle $\pi/4$ with vertices at ∂D and axes parallel to ox_3. Then $x_3 = \lambda_0(x_1, x_2)$ is the envelope of this family of cones located above D. When $\psi^\circ = |\mathbf{e}|$, the problem of constructing λ_0 and the properties of λ_0 when ∂D is a Jordan curve have been examined in [7]. If $\psi^\circ(x_1, x_2, \mathbf{e}) = m(\mathbf{e})$, then the scheme to construct λ_0 is the same, only circular cones must be replaced by Monge cones [8].

b) Let $\psi^\circ(x_1, x_2, \mathbf{e}) = a(x_1, x_2)|\mathbf{e}|$, $a(x_1, x_2) > c > 0$ and let the function $a(x_1, x_2)$ be constant on the level lines of the function $\lambda_0^1(x_1, x_2)$, constructed in D for $\psi^\circ(x_1, x_2, \mathbf{e}) = |\mathbf{e}|$. In this case $\lambda_0(x_1, x_2)$ has the same level lines as λ_0^1 and $a(x_1, x_2)\,\partial\lambda_0/\partial n = 1$, where $\partial/\partial n$ denotes differentiation in a direction orthogonal to the level lines λ_0^1. The warping rate in this case is evidently exactly the same as in case (a) [9] although the quantity c_* may be different.

The class of functions $a(x_1, x_2)$ can be expanded somewhat by taking into account that the conformal mapping $\xi_i = \xi_i(x_1, x_2)$, $i = 1,2$ transfers $\lambda_0(x_1, x_2)$ into the function $\mu_0(\xi_1, \xi_2)$ which satisfies the equation $a|\nabla_x \xi_1|\nabla\xi\,\mu_0| = 1$ almost everywhere in D_ξ (D_ξ is the image of D under the mapping ξ).

c) Let $\psi^\circ(x_1, x_2, \mathbf{e})$ be a function of general form. Let us construct a family of Monge cones at the points ∂D and let us take their envelope. We consider a level line of height ε on this envelope and we construct a second family of Monge cones with vertices at points of this line at a height ε above the plane D. Then the level line at the height 2ε is considered the envelope of the second family of Monge cones. A third family of Monge cones with vertices at the height 2ε is constructed at points of this level line, etc. Let us define $\lambda_\varepsilon(x_1, x_2)$ as the envelope of the first family of Monge cones in the subdomain D, where $0 \leqslant \lambda_\varepsilon \leqslant \varepsilon$, in the subdomain D where $\varepsilon \leqslant \lambda_\varepsilon \leqslant 2\varepsilon$, the function $\lambda_\varepsilon(x_1, x_2)$ is the envelope of the second family of Monge cones, etc. As $\varepsilon \to 0$ the functions λ_ε converge to the required function λ_0.

In the case of multiconnected domains D (we consider the case of a doubly-connected domain), the function λ_0 is determined as follows. We examine the simply-connected domain Ω bounded by the contour Γ_1, and we construct the stress function λ_0^1 therein. We draw the level line Γ: $\lambda_0^1 = c$ such that Γ_2 is within Γ and Γ, Γ_2 have common points. Then, the stress function $\lambda_0^2(x_1, x_2)$ is constructed in the simply-connected domain bounded by Γ, Γ_2 The required function λ_0 agrees with λ_0^1 in the ring-shaped domain bounded by Γ_1, Γ and agrees with $\lambda_0^2 + c$ in the simply-connected domain bounded by the curves Γ_2, Γ_1.

5. Survey of investigations on the torsion of rigid-plastic bars. The solution of the problem of torsion of rigid-plastic bars has been elucidated in many monographs, surveys, and reference texts on the theory of plasticity (see [1, 10 — 14], for instance, where references to the journal literature are also presented). The solution proposed in [1, 10 — 14] for the torsion problem is the following. A special tensor field

$\sigma_{13}^\circ = \partial\lambda_0 / \partial x_2$, $\sigma_{23}^\circ = -\partial\lambda_0 / \partial x_1$ is considered, the remaining σ_{ij}° are zero, where $\lambda_0(x_1, x_2)$ is such that $\lambda_0|_{\partial D} = c$ and $|\nabla\lambda_0| = k(x_1, x_2)$ almost everywhere in D (only isotropic inhomogeneous media were considered) and λ_0 yields the upper bound to the functional $\Phi(\lambda) = -\int_D \mathbf{t}_0 G\lambda \, d\mu \left(\int_D (x_1^2 + x_2^2) \, d\mu\right)^{-1}$, $\sup_\lambda \Phi(\lambda) + \Phi(\lambda_0)$

It is asserted that $\Phi(\lambda_0)$ is the limit moment.

It has been shown in Sect. 1 of this paper that $\Phi(\lambda_0)$ generally yields the lower bound of the limit moment. It is proved in Sect. 2 that $\Phi(\lambda_0)$ is the limit moment.

Another means of obtaining this result (this means is considered in the literature cited and is realized in Sect. 3 of the paper) is to find the velocity field corresponding to the special stress field, which is equivalent to the solvability of the over-defined system (3.2) which has the form

$$\frac{k(\partial u / \partial x_1 - x_2)}{|\nabla u - \mathbf{t}_0|} = \frac{\partial\lambda_0}{\partial x_2}, \qquad \frac{k(\partial u / \partial x_2 + x_1)}{|\nabla u - \mathbf{t}_0|} = -\frac{\partial\lambda_0}{\partial x_1} \tag{5.1}$$

in the case under consideration.

However, not this system but its corollary

$$\frac{\partial u}{\partial x_1}\frac{\partial\lambda_0}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial\lambda_0}{\partial x_2} + x_1\frac{\partial\lambda_0}{\partial x_2} - x_2\frac{\partial\lambda_0}{\partial x_1} = 0 \tag{5.2}$$

is later investigated in the literature cited above. It is clear that (5.2) is not equivalent to the system (5.1).

The lack of equivalence can be detected in the simplest examples. Let D be a circular concentric ring with inner radius r_- and outer radius r_+. Then the general solution of (5.2) has the form $u = c(\theta)$, where θ is the polar angle with pole at the center of the ring and $c(\theta)$ is an arbitrary function. At the same time, the general solution of the system (5.1) is

$$u = -r_-^2\theta + k(\theta), \qquad k'(\theta) \geqslant 0, \qquad k(2\pi) - k(0) \leqslant r_-^2 2\pi$$

Certainly, (5.2) can be solved in specific problems and it will be seen by direct substitution that the relations (5.1) are satisfied. It is apparently easy to perform such a confirmation when the solution of (5.2) is given in explicit form and is continuous. Such formulas for several simply-connected domains (a special kind of oval, equilateral triangle, rectangle, corner) in the case of a homogeneous isotropic medium are presented in [12]. The corresponding formula for the warping rate of the cross section of a rectangular cylinder for a special kind of inhomogeneous isotropic medium is contained in [9].

In the case of doubly-connected cross sections, the warping rate is generally a discontinuous function, as has been shown in Sect. 3, and it is here impossible to replace the investigation of the system (5.1) by an investigation of (5.2). Moreover, the question of the uniqueness of the solution remained open in the previous exposition of the problem. Let us note that (3.12) for the warping rate has not been known before even in the case of a homogeneous, isotropic rigid-plastic medium.

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